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TECHNICAL TRANSLATION

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RELATIVISTIC ROCKET MECHANICS

By H. G. L. Krause

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RELATIVISTIC ROCKET MECHANICS*

By H. G. L. Krause**

Summary

In this work the mechanics of the Special Theory of Relativity are extended to systems having a rest mass changing in time (rockets). The principles of momentum and energy are investigated as well as the law of the decrease of mass of an arbitrarily accelerated rocket in free space without external forces, and in fact in the system of the stationary observer on earth and in the system of the astronaut moving with the rocket. Then two special cases are treated; namely, the motion of a rocket with constant self-acceleration and the motion of a rocket with constant thrust.

I. Derivation of the Relation between Acceleration and Self-acceleration of a Rocket from the Lorentz Transformations

If one considers a stationary co-ordinate system $K\,[\mathfrak{r}(x, y, z); t]$ and a uniformly moving system $K'[\mathfrak{r}'(x', y', z'); t']$, which is moving with a constant velocity $\mathfrak{v}$ parallel to the system K (velocity of the system), then the transition from one system to the other is given by the general

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Lorentz transformation [1]:

$$\mathfrak{r}' = \mathfrak{r} - \mathfrak{v}\left\{\frac{(\mathfrak{r}\,\mathfrak{v})}{v^2}\left(1-\frac{1}{a}\right)+\frac{t}{a}\right\}; \qquad t' = \frac{t - \dfrac{(\mathfrak{r}\,\mathfrak{v})}{c^2}}{a} \tag{1}$$

or conversely

$$\mathfrak{r} = \mathfrak{r}' + \mathfrak{v}\left\{-\frac{(\mathfrak{r}'\,\mathfrak{v})}{v^2}\left(1-\frac{1}{a}\right)+\frac{t'}{a}\right\}; \qquad t = \frac{t' + \dfrac{(\mathfrak{r}'\,\mathfrak{v})}{c^2}}{a} \tag{2}$$

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with the abbreviations (c = velocity of light)

$$\beta^2 = \frac{v^2}{c^2}, \qquad a = \sqrt{1-\beta^2}, \qquad \beta = \sqrt{1-a^2}.$$

In this connection $\mathfrak{r} = \mathfrak{r}' = 0$ for $t = t' = 0$. If a point has the velocity $\mathfrak{w} = \dot{\mathfrak{r}}$ relative to the system K and the velocity $\mathfrak{w}' = \dot{\mathfrak{r}}'$ relative to the system K', then from differentiation with respect to time of Equations (1) and (2) there follow the relations:

$$\mathfrak{w}' = \frac{d\mathfrak{r}'}{dt'} = \frac{d\mathfrak{r}'/dt}{dt'/dt} = \frac{\mathfrak{w}\,a - \mathfrak{v}\left\{\dfrac{(\mathfrak{w}\,\mathfrak{v})}{v^2}(a-1)+1\right\}}{1-\dfrac{(\mathfrak{w}\,\mathfrak{v})}{c^2}} \tag{3}$$

and

$$\mathfrak{w} = \frac{d\mathfrak{r}}{dt} = \frac{d\mathfrak{r}/dt'}{dt/dt'} = \frac{\mathfrak{w}'\,a + \mathfrak{v}\left\{-\dfrac{(\mathfrak{w}'\,\mathfrak{v})}{v^2}(a-1)+1\right\}}{1+\dfrac{(\mathfrak{w}'\,\mathfrak{v})}{c^2}} \tag{4}$$

as foundation for the relativistic kinematics, in which

$$\frac{dt'}{dt} = \frac{1}{a}\left\{1-\frac{(\mathfrak{w}\,\mathfrak{v})}{c^2}\right\} \tag{5}$$

If $\mathfrak{w}\,\|\,\mathfrak{v}$, , or $\mathfrak{w}'\,\|\,\mathfrak{v}$, then the addition theorem for velocities reads

$$\mathfrak{w}' = \frac{\mathfrak{w}-\mathfrak{v}}{1-\dfrac{(\mathfrak{w}\,\mathfrak{v})}{c^2}}; \qquad |\mathfrak{w}'| = c\left[1-\frac{\left(1-\left|\dfrac{\mathfrak{w}}{c}\right|\right)\cdot\left(1+\left|\dfrac{\mathfrak{v}}{c}\right|\right)}{1-\dfrac{(\mathfrak{w}\,\mathfrak{v})}{c^2}}\right] \leqq c \tag{6}$$

or, respectively,

$$\mathfrak{w} = \frac{\mathfrak{w}'+\mathfrak{v}}{1+\dfrac{(\mathfrak{w}'\,\mathfrak{v})}{c^2}}; \qquad |\mathfrak{w}| = c\left[1-\frac{\left(1-\left|\dfrac{\mathfrak{v}'}{c}\right|\right)\cdot\left(1-\left|\dfrac{\mathfrak{v}}{c}\right|\right)}{1+\dfrac{(\mathfrak{w}'\,\mathfrak{v})}{c^2}}\right] \leqq c. \tag{7}$$

For relativistic dynamics one needs the accelerations, which are obtained by differentiating Equations (3) and (4) with respect to time:

$$b' = \frac{d\mathfrak{w}'}{dt'} = \frac{d\mathfrak{w}'/dt}{dt'/dt} = a^2 \frac{\dot{\mathfrak{w}}\left(1 - \frac{(\mathfrak{w}\,\mathfrak{v})}{c^2}\right) + \frac{(\dot{\mathfrak{w}}\,\mathfrak{v})}{c^2}\left\{\mathfrak{w} - \frac{\mathfrak{v}}{1+a}\right\}}{\left(1 - \frac{(\mathfrak{w}\,\mathfrak{v})}{c^2}\right)^3} \tag{8}$$

or respectively,

$$b = \dot{w} = \frac{d\mathfrak{w}}{dt} = \frac{d\mathfrak{w}/dt'}{dt/dt'} = a^2 \frac{\dot{\mathfrak{w}}'\left(1 + \frac{(\mathfrak{w}'\,\mathfrak{v})}{c^2}\right) - \frac{(\dot{\mathfrak{w}}'\,\mathfrak{v})}{c^2}\left\{\mathfrak{w}' + \frac{\mathfrak{v}}{1+a}\right\}}{\left(1 + \frac{(\mathfrak{w}'\,\mathfrak{v})}{c^2}\right)^3}. \tag{9}$$

These equations are now to be applied to the motion of a fast rocket. At time t let the rocket have the speed $\mathfrak{v}$ relative to the system K of the stationary observer on Earth (e.g. at the position of origin). Let the starting point of the moving system K′ of the astronaut now be taken in the rocket (space ship occupants). The system velocity with which the system K′ moves relative to the system K is therefore equal to the velocity $\mathfrak{v}$ of the rocket in the system K. If $\mathfrak{w} = \mathfrak{v}$ is inserted into Equation (3), then we obtain $\mathfrak{w}' = 0$. (transformation to stationary state). In the system K′ of the astronaut the velocity of the rocket is therefore $\mathfrak{w}' = 0$. Furthermore, from Eq. (5) there follows

$$\frac{dt'}{dt} = \sqrt{1 - \frac{v^2}{c^2}} = \sqrt{1 - \beta^2} = a. \tag{10}$$

The time t′ belonging to the system K′ is also called the local time or the proper time τ. The acceleration in the stationary system K′ is yielded if $\mathfrak{w}$ is set $= \mathfrak{v}$ in Eq. (8), that is

$$b' = \dot{\mathfrak{w}}' = \frac{a\,\dot{\mathfrak{w}} + \frac{(\dot{\mathfrak{w}}\,\mathfrak{v})}{c^2}\cdot\frac{\mathfrak{v}}{1+a}}{a^3} = \frac{a\,\dot{\mathfrak{w}} + \frac{(\dot{\mathfrak{w}}\,\mathfrak{v})}{v^2}\,\mathfrak{v}\,(1-a)}{a^3}. \tag{11}$$

If $\dot{\mathfrak{w}} \parallel \mathfrak{v}$, then it follows that

$$b' = \dot{\mathfrak{w}}' = \frac{\dot{\mathfrak{w}}}{a^3} = \frac{\dot{\mathfrak{w}}}{(1-\beta^2)^{3/2}}; \tag{12}$$

whereas for $\dot{\mathfrak{w}} \perp \mathfrak{v}$

$$b' = \dot{\mathfrak{w}}' = \frac{\dot{\mathfrak{w}}}{a^2} = \frac{\dot{\mathfrak{w}}}{1-\beta^2}. \tag{13}$$

Since the system of the astronaut is coupled to the rocket, $\mathfrak{v}$ and $a = \sqrt{1 - v^2/c^2}$ are now functions of the time t. Consequently, with $\mathfrak{w} = \mathfrak{v}$, it now turns out:

If $\dot{\mathfrak{w}} \parallel \mathfrak{v}$, then with regard to Eqs. (14) and (12) for an accelerated system the following is true:

$$\frac{d}{dt}\left(\frac{\mathfrak{w}}{\sqrt{1 - v^2/c^2}}\right) \equiv \frac{d}{dt}\left(\frac{\mathfrak{w}}{a}\right) = \frac{\dot{\mathfrak{w}}}{a^3} = b' \equiv \dot{\mathfrak{w}}'. \tag{15}$$

But for $\dot{\mathfrak{w}} \perp \mathfrak{v}$, from Eqs. (14) and (13), there follows:

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$$\frac{d}{dt}\left(\frac{\mathfrak{w}}{\sqrt{1 - v^2/c^2}}\right) \equiv \frac{d}{dt}\left(\frac{\mathfrak{w}}{a}\right) = \frac{\dot{\mathfrak{w}}}{a} = a\, b' \equiv a\, \dot{\mathfrak{w}}'. \tag{16}$$

II. The Thrust and the Law of the Decrease of Mass of an Arbitrarily Accelerated Rocket in Free Space without External Forces

In the following the data for the rocket and for the discharged gases in the system K of the stationary earth observer (e.g. at the starting place) are to be marked without an index and those in the system K_0 of the astronaut moving with the rocket with the index 0, as is done in Table 1.

Table 1

	In system K of stationary earth observer	In system K_0 of moving astronaut
Velocity of rocket	v	0
Actual mass of rocket	$m = \dfrac{m_0}{\sqrt{1 - v^2/c^2}}$	m_0
Velocity of discharged gases	$a = \dfrac{a_0 - v}{1 - \dfrac{a_0 v}{c^2}}$	a_0
Mass of discharged gases	$dm = \dfrac{dm_0^*}{\sqrt{1 - a^2/c^2}}$	$dm_0 = \dfrac{dm_0^*}{\sqrt{1 - a_0^2/c^2}}$
Element of time	$dt = \dfrac{dt_0}{\sqrt{1 - v^2/c^2}}$	dt_0
Acceleration of rocket	$b = b_0\,(1 - v^2/c^2)^{3/2}$	b_0
Force of thrust of rocket	$\dfrac{d}{dt}(m\,v) = -\,a\,\dfrac{dm}{dt} = F$	$m_0\, b_0 = -\,a_0\,\dfrac{dm_0}{dt_0} = F_0$

If m_0 is the rest mass of the rocket and dm_0^* the rest mass of the discharged gases, then the mass of the rocket and respectively the mass of the gases in the system K of the earth observer are:

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}, \qquad \text{, respectively} \qquad dm = \frac{dm_0^*}{\sqrt{1 - a^2/c^2}}. \tag{17}$$

F 3 6 In this system the momentum principle reads

$$\frac{d}{dt}\left(\frac{m_0\, \mathfrak{v}}{\sqrt{1 - v^2/c^2}}\right) = -\,\mathfrak{a}\,\frac{dm_0^*/dt}{\sqrt{1 - a^2/c^2}} \equiv \mathfrak{F} \tag{18}$$

or in view of Eq. (17)

$$\frac{d}{dt}(m\, \mathfrak{v}) = -\,\mathfrak{a}\,\frac{dm}{dt} \equiv \mathfrak{F}. \tag{19}$$

In this connection $\mathfrak{F}$ is the force of thrust of the rocket in the system of the earth observer. From application of the relativistic addition theorem for velocities we obtain, if the thrust acts in the direction of the velocity $\mathfrak{v}$ $(\mathfrak{a} \| \mathfrak{v})$:

$$\mathfrak{a} = \frac{\mathfrak{a}_0 - \mathfrak{v}}{1 - \frac{(\mathfrak{a}_0\, \mathfrak{v})}{c^2}}, \qquad \text{or conversely,} \qquad \mathfrak{a}_0 = \frac{\mathfrak{a} + \mathfrak{v}}{1 + \frac{(\mathfrak{a}\, \mathfrak{v})}{c^2}}, \tag{20}$$

thus

$$d(m\, \mathfrak{v}) = m\, d\mathfrak{v} + \mathfrak{v}\, dm = -\,\mathfrak{a}\, dm = -\,\frac{\mathfrak{a}_0 - \mathfrak{v}}{1 - \frac{(\mathfrak{a}_0\, \mathfrak{v})}{c^2}}\, dm$$

or

$$m\, d\mathfrak{v} = -\,(\mathfrak{a} + \mathfrak{v})\, dm = -\left(\frac{\mathfrak{a}_0 - \mathfrak{v}}{1 - \frac{(\mathfrak{a}_0\, \mathfrak{v})}{c^2}} + \mathfrak{v}\right) dm = -\,\mathfrak{a}_0\,\frac{1 - \frac{v^2}{c^2}}{1 - \frac{(\mathfrak{a}_0\, \mathfrak{v})}{c^2}}\, dm.$$

Therewith we have

$$\frac{dm}{m} = -\,\frac{1 - \frac{(\mathfrak{a}_0\, \mathfrak{v})}{c^2}}{a_0\,(1 - v^2/c^2)}\, dv. \tag{21}$$

Now, because of Eq. (17)

$$\frac{dm}{m} = \frac{\sqrt{1 - v^2/c^2}}{m_0}\, d\left(\frac{m_0}{\sqrt{1 - v^2/c^2}}\right) = \frac{dm_0}{m_0} + \sqrt{1 - \frac{v^2}{c^2}}\, d\left(\frac{1}{\sqrt{1 - v^2/c^2}}\right) =$$

$$= \frac{dm_0}{m_0} + \frac{\frac{(\mathfrak{v}\, d\mathfrak{v})}{c^2}}{1 - v^2/c^2} = \frac{dm_0}{m_0} - \frac{\frac{(\mathfrak{a}_0\, \mathfrak{v})}{c^2}\, c\, v}{a_0\,(1 - v^2/c^2)}. \tag{22}$$

If Eq. (21) is inserted into Eq. (22), there follows immediately $(\mathfrak{a}_0 \| \mathfrak{v})$ F 3 6

$$\frac{dm_0}{m_0} = -\frac{dv}{a_0\,(1 - v^2/c^2)}. \tag{23}$$

For constant a_0 this equation can be easily integrated. If M_0 is the rest mass of the rocket at time t=0 (v =0), there results

$$\ln \frac{m_0}{M_0} = -\frac{c}{a_0} \int\limits_0^v \frac{\frac{1}{c}\, dv}{1 - \left(\frac{v}{c}\right)^2} = -\frac{c}{2\, a_0} \ln \frac{1 + v/c}{1 - v/c}$$

or

$$\frac{m_0}{M_0} = \left[\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}\right]^{\frac{c}{2a_0}}. \tag{24}$$

or conversely,

$$\frac{v}{c} = \frac{1 - \left(\frac{m_0}{M_0}\right)^{\frac{2a_0}{c}}}{1 + \left(\frac{m_0}{M_0}\right)^{\frac{2a_0}{c}}}. \tag{25}$$

The relation (24), found by J. Ackeret [2] (1945) can be transformed by means of the relations $\beta = v/c = \sqrt{1 - \alpha^2}$ or $\alpha = \sqrt{1 - v^2/c^2} = \sqrt{1 - \beta^2}$ into

$$\frac{m_0}{M_0} = \left[\frac{1 - \beta}{1 + \beta}\right]^{\frac{c}{2a_0}} = \left[\frac{\sqrt{1 - \beta^2}}{1 + \beta}\right]^{\frac{c}{a_0}} = \left[\frac{\alpha}{1 + \sqrt{1 - \alpha^2}}\right]^{\frac{c}{a_0}}. \tag{26}$$

The relativistic basic equation for the rocket (24) becomes in classical physics ($v/c \to 0$) the well-known formula:

$$r \equiv \frac{M_0}{m_0} = e^{\frac{v}{a_0}} \qquad \text{viz.} \qquad \ln r = \frac{v}{a_0},$$

since

$$\lim_{\frac{v}{c}\to 0} \ln r = \lim_{\frac{v}{c}\to 0} \frac{v}{a_0} \cdot \frac{\ln\left(1+\frac{v}{c}\right) - \ln\left(1-\frac{v}{c}\right)}{2\frac{v}{c}} = \lim_{\frac{v}{c}\to 0} \frac{v}{a_0} \frac{\frac{1}{1+v/c} + \frac{1}{1-v/c}}{2} = \frac{v}{a_0}$$

F 3 6 If one sets $x = v/c$ $(0 \leqq x \leqq 1)$ 1) and $y = 1/r = m_0/M_0$ $(0 \leqq y \leqq 1)$, then according to Eq. (24):

$$y = \left[\frac{1-x}{1+x}\right]^{\frac{c}{2a_0}}, \qquad y' = -\frac{c}{a_0(1+x)^2}\left[\frac{1-x}{1+x}\right]^{\frac{c}{2a_0}-1}$$

$$y'' = \frac{2\frac{c}{a_0}}{(1+x)^4}\left[\frac{1-x}{1+x}\right]^{\frac{c}{2a_0}-2} \cdot \left\{\frac{c}{2a_0} - x\right\}.$$

An inflection point ($y''=0$) appears for

$$x_w = \frac{c}{2a_0} \qquad \text{and} \qquad y_w = \left[\frac{1-\frac{c}{2a_0}}{1+\frac{c}{2a_0}}\right]^{\frac{c}{2a_0}} = \left[\frac{1-x_w}{1+x_w}\right]^{x_w},$$

in case $a_0 > c/2$ holds. In the most favorable case $a_0 = c$ (photons) we get $x_w = 1/2$ and $y_w = 1/\sqrt{3}$, that is the inflection point appears when $v = c/2$ and $r = \sqrt{3}$.

Because of relation (20) there follow the relations

$$\left(1 + \frac{(\mathfrak{a}\,\mathfrak{v})}{c^2}\right) \cdot \left(1 - \frac{(\mathfrak{a}_0\,\mathfrak{v})}{c^2}\right) = 1 - \frac{v^2}{c^2}; \qquad \left(1 + \left|\frac{\mathfrak{v}}{\mathfrak{a}}\right|\right) \cdot \left(1 - \left|\frac{\mathfrak{v}}{\mathfrak{a}_0}\right|\right) = 1 - \frac{v^2}{c^2} \tag{27}$$

and

$$\frac{\sqrt{1 - a_0^2/c^2}}{\sqrt{1 - a^2/c^2}} = \frac{1 - \frac{(\mathfrak{a}_0\,\mathfrak{v})}{c^2}}{\sqrt{1 - v^2/c^2}} = \frac{\sqrt{1 - v^2/c^2}}{1 + \frac{(\mathfrak{a}\,\mathfrak{v})}{c^2}}. \tag{28}$$

A comparison of Eq. (21) and (22) indicates that

$$\frac{dm}{m} = \left[1 - \frac{(\mathfrak{a}_0\,\mathfrak{v})}{c^2}\right]\frac{dm_0}{m_0}; \tag{29}$$

thus because of Eq. (28) and (17)

$$\frac{dm}{dm_0} = \left[1 - \frac{(\mathfrak{a}_0\,\mathfrak{v})}{c^2}\right]\frac{m}{m_0} = \frac{1 - \frac{(\mathfrak{a}_0\,\mathfrak{v})}{c^2}}{\sqrt{1 - v^2/c^2}} = \frac{\sqrt{1 - a_0{}^2/c^2}}{\sqrt{1 - a^2/c^2}} \qquad \textbf{(30)}$$

and with reference to Eq. (17) F 3 6

$$dm_0 = \frac{dm_0{}^*}{\sqrt{1 - a_0{}^2/c^2}}. \qquad \textbf{(31)}$$

This is thus the mass of the discharged gases in the system K_0 of the astronaut traveling with the rocket. The thrust force $\mathfrak{F}_0$ in this system should be derived now. According to Eq. (10)

$$dt_0 = dt\sqrt{1 - v^2/c^2} \qquad \textbf{(32)}$$

and,. in case the acceleration $\mathfrak{b} = \dot{\mathfrak{v}} = \ddot{\mathfrak{r}}$ acts in the direction of the velocity $\mathfrak{v}$, according to Eq. (15)

$$\frac{d}{dt}\left(\frac{\mathfrak{v}}{\sqrt{1 - v^2/c^2}}\right) = \frac{\dot{\mathfrak{v}}}{(1 - v^2/c^2)^{3/2}} = \frac{\mathfrak{b}}{(1 - v^2/c^2)^{3/2}} = \mathfrak{b}_0. \qquad \textbf{(33)}$$

Upon application of both of these last relationships Eq. (18) then yields

$$\mathfrak{F} = \frac{d}{dt}\left(\frac{m_0\,\mathfrak{v}}{\sqrt{1 - v^2/c^2}}\right) = \frac{m_0\,\dot{\mathfrak{v}}}{(1 - v^2/c^2)^{3/2}} + \frac{\mathfrak{v}}{\sqrt{1 - v^2/c^2}}\frac{dm_0}{dt} = m_0\,\mathfrak{b}_0 + \mathfrak{v}\frac{dm_0}{dt_0}. \qquad \textbf{(34)}$$

On the other hand, according to Equations (19), (30), (20), and (32):

$$\mathfrak{F} = -\mathfrak{a}\frac{dm}{dt} = -\mathfrak{a}\frac{1 - \frac{(\mathfrak{a}_0\,\mathfrak{v})}{c^2}}{\sqrt{1 - v^2/c^2}}\frac{dm_0}{dt} = -(\mathfrak{a}_0 - \mathfrak{v})\frac{dm_0}{dt_0} = -\mathfrak{a}_0\frac{dm_0}{dt_0} + \mathfrak{v}\frac{dm_0}{dt_0}. \qquad \textbf{(35)}$$

A comparison of the two Eqs. (34) and (35) yields for the thrust force $\mathfrak{F}_0$ of the rocket in the system of the astronaut (space ship occupants)

$$\frac{m_0\,\dot{\mathfrak{v}}}{(1 - v^2/c^2)^{3/2}} = m_0\,\mathfrak{b}_0 = \mathfrak{F}_0 = -\mathfrak{a}_0\frac{dm_0}{dt_0} = -\mathfrak{a}_0\frac{dm_0{}^*/dt_0}{\sqrt{1 - a_0{}^2/c^2}}. \qquad \textbf{(36)}$$

Furthermore, there follows from Eqs. (35) and (36)

$$\mathfrak{F} = \frac{\mathfrak{a}_0 - \mathfrak{v}}{\mathfrak{a}_0}\,\mathfrak{F}_0 = \left(1 - \frac{v}{a_0}\right)\mathfrak{F}_0. \tag{37}$$

If $v = a_0$ is thereby attained, then a becomes $= 0$ and likewise also $\mathfrak{F} = 0$. From the technical point of view the primary thrust force $\mathfrak{F}_0$ of the rocket in the stationary system K_0 is naturally the matter of primary interest. If we transform Eq. (36) in the following manner:

$$\frac{dm_0}{m_0} = -\frac{b_0}{a_0}\,dt_0 = -\frac{\dot{v}}{a_0\,(1 - v^2/c^2)^{3/2}}\,dt_0 = -\frac{dv}{a_0\,(1 - v^2/c^2)},$$

then we obtain Eq. (23) again. From Eqs. (36) and (37) there follow further the relations

$$\mathfrak{F}_0 = m_0\,b_0 = \frac{m_0\,\dot{\mathfrak{v}}}{(1 - v^2/c^2)^{3/2}} = \frac{m\,\dot{\mathfrak{v}}}{1 - v^2/c^2} = \frac{m\,\dot{b}}{1 - v^2/c^2}$$

and

$$\mathfrak{F} = \left(1 - \frac{v}{a_0}\right)\cdot\mathfrak{F}_0 = \frac{1 - \dfrac{v}{a_0}}{1 - v^2/c^2}\,m\,\dot{\mathfrak{v}} = \frac{m\,b}{1 + v/a}.$$

That is

$$\mathfrak{F} < m\,\dot{\mathfrak{v}} < \mathfrak{F}_0.$$

Finally there should still be derived the energy principle.

$$E_0 = m_0\,c^2 \tag{38}$$

is the total energy of the rocket in the system K_0 of the astronaut and

$$E = m\,c^2 = \frac{m_0\,c^2}{\sqrt{1 - v^2/c^2}} = \frac{E_0}{\sqrt{1 - v^2/c^2}} \tag{39}$$

that in the system K of the earth observer. From Eqs. (30) and (32) there results

$$\frac{dm}{dt} = \left(1 - \frac{(\mathfrak{a}_0\,\mathfrak{v})}{c^2}\right)\frac{dm_0}{dt_0}, \tag{40}$$

accordingly also

$$\frac{dE}{dt} = \left(1 - \frac{(\mathfrak{a}_0\,\mathfrak{v})}{c^2}\right)\frac{dE_0}{dt_0}, \tag{41}$$

or because of Eq. (36) also

$$\frac{dE}{dt} = \frac{dE_0}{dt_0} - (\mathfrak{a}_0\,\mathfrak{v})\frac{dm_0}{dt_0} = \frac{dE_0}{dt_0} + (\mathfrak{v}\,\mathfrak{F}_0). \tag{42}$$

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The energy principle can, according to that, be written:

$$(\mathfrak{v}\,\mathfrak{F}_0) = \frac{dE}{dt} - \frac{dE_0}{dt_0} = \frac{dE_{kin}}{dt} - \left(1 - \sqrt{1 - v^2/c^2}\right)\frac{dE_0}{dt_0}; \tag{43}$$

in connection with this

$$\begin{aligned} E_{kin} &= E - E_0 = (m - m_0)\,c^2 = m\,c^2\left(1 - \sqrt{1 - v^2/c^2}\right) = \\ &= m_0\,c^2\left(\frac{1}{\sqrt{1 - v^2/c^2}} - 1\right) = \frac{m_0\,v^2}{2}\left(1 + \frac{3}{4}\frac{v^2}{c^2} + \dots\right) \end{aligned} \tag{44}$$

is the kinetic energy. The energy principle thus signifies that the work done by the force of the thrust per unit of time is equal to the change in energy of the system per unit of time. For systems having a rest mass constant in time the energy principle in the well-known form follows immediately

$$(\mathfrak{v}\,\mathfrak{F}) = \frac{dE}{dt} = \frac{dE_{kin}}{dt}.$$

The generalized energy principle (43) can, in view of Eq. (34), also be written in the following manner:

$$\begin{aligned} (\mathfrak{v}\,\mathfrak{F}) &= (\mathfrak{v}\,\mathfrak{F}_0) + v^2\frac{dm_0}{dt_0} = (\mathfrak{v}\,\mathfrak{F}_0) + \frac{v^2}{c^2}\frac{dE_0}{dt_0} = \\ &= \frac{dE}{dt} - \left(1 - \frac{v^2}{c^2}\right)\frac{dE_0}{dt_0} = \frac{dE}{dt} - \sqrt{1 - v^2/c^2}\,\frac{dE_0}{dt}; \end{aligned}$$

III. Motion and Mass Consumption of a Rocket With Constant Self-acceleration

This case was first, even if not exhaustively, treated by R. Esnault-Pelterie [3]. A short contribution to this problem was also made by W. L. Bade [4]. For a rocket which travels with a constant self-acceleration b_0, upon applying Eq. (33), there holds true

$$\frac{d}{dt}\left(\frac{v}{\sqrt{1-v^2/c^2}}\right)=\frac{\dot v}{(1-v^2/c^2)^{3/2}}=b_0\ (\text{const.});\tag{46}$$

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thus

$$v=b_0 t\sqrt{1-v^2/c^2}=\alpha\, b_0 t\quad (\text{with}\ \ v=0\ \ \text{for}\ \ t=0)$$

that is

$$\frac{1}{b_0 t}=\sqrt{\frac{1}{v^2}-\frac{1}{c^2}}$$

and consequently the velocity

$$v=\dot x=\frac{1}{\sqrt{\dfrac{1}{b_0^2 t^2}+\dfrac{1}{c^2}}}=c\,\frac{\dfrac{b_0}{c}t}{\sqrt{1+\left(\dfrac{b_0}{c}t\right)^2}}=b_0 t\left[1-\frac{1}{2}\left(\frac{b_0}{c}t\right)^2+\frac{1\cdot 3}{2\cdot 4}\left(\frac{b_0}{c}t\right)^4\mp\dots\right].\tag{47}$$

Hence one gets

$$\alpha=\sqrt{1-\frac{v^2}{c^2}}=\frac{v}{b_0 t}=\frac{1}{\sqrt{1+\left(\dfrac{b_0}{c}t\right)^2}}=1-\frac{1}{2}\left(\frac{b_0}{c}t\right)^2+\frac{1\cdot 3}{2\cdot 4}\left(\frac{b_0}{c}t\right)^4\mp\dots$$

and for the acceleration

$$b=\dot v=\ddot x=b_0\alpha^3=\frac{b_0}{\left[1+\left(\dfrac{b_0}{c}t\right)^2\right]^{3/2}}=$$

$$=b_0\left[1-\frac{1\cdot 3}{1}\,\frac{1}{2}\left(\frac{b_0}{c}t\right)^2+\frac{1\cdot 3\cdot 5}{1\cdot 2}\,\frac{1}{2^2}\left(\frac{b_0}{c}t\right)^4\mp\dots\right].\tag{49}$$

The inverse of Eq. (47) is now

$$\frac{b_0}{c}t=\frac{v/c}{\alpha}=\frac{\beta}{\alpha}=\frac{\sqrt{1-\alpha^2}}{\alpha}=\frac{v/c}{\sqrt{1-v^2/c^2}}.\tag{50}$$

From the integration of Eq. (47) there follows for the path (displacement)

$$x = \int_0^t v\,dt = \frac{c^2}{b_0}\int_0^t \frac{\frac{b_0}{c}t}{\sqrt{1+\left(\frac{b_0}{c}t\right)^2}}\,\frac{b_0}{c}\,dt = \frac{c^2}{b_0}\int_0^{\frac{b_0}{c}t} \frac{z\,dz}{\sqrt{1+z^2}} = \frac{c^2}{b_0}\sqrt{1+z^2}\,\Big|^{\frac{b_0}{c}t},$$

therefore

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$$x = \frac{c^2}{b_0}\left[\sqrt{1+\left(\frac{b_0}{c}t\right)^2} - 1\right] = \frac{b_0}{2}t^2\left[1 - \frac{1}{4}\left(\frac{b_0}{c}t\right)^2 + \frac{1\cdot 3}{4\cdot 6}\left(\frac{b_0}{c}t\right)^4 \mp \dots\right]. \tag{51}$$

The inverse of this relation reads

$$t = \frac{c}{b_0}\sqrt{\left(1+\frac{b_0 x}{c^2}\right)^2 - 1} = \sqrt{\frac{2x}{b_0} + \left(\frac{x}{c}\right)^2}. \tag{52}$$

If we measure the time not in the system of the stationary earth observer but in the system of the astronaut traveling along with the rocket (proper or local time t_0), then according to Eqs. (32) and (48) we have the following relationship:

$$t_0 = \int_0^t \sqrt{1 - v^2/c^2}\,dt =$$

$$= \frac{c}{b_0}\int_0^t \frac{\frac{b_0}{c}\,dt}{\sqrt{1+\left(\frac{b_0}{c}t\right)^2}} = \frac{c}{b_0}\int_0^{\frac{b_0}{c}t} \frac{dz}{\sqrt{1+z^2}} = \frac{c}{b_0}\ln\left(z+\sqrt{1+z^2}\right)\Big|_0^{\frac{b_0}{c}t},$$

thus

$$\begin{aligned} t_0 &= \frac{c}{b_0}\ln\left[\frac{b_0}{c}t + \sqrt{1+\left(\frac{b_0}{c}t\right)^2}\right] = \frac{c}{b_0}\,\mathrm{Ar\,Sin}\left(\frac{b_0}{c}t\right) = \\ &= t\left[1 - \frac{1}{2\cdot 3}\left(\frac{b_0}{c}t\right)^2 + \frac{1\cdot 3}{2\cdot 4\cdot 5}\left(\frac{b_0}{c}t\right)^4 \mp \dots\right]. \end{aligned} \tag{53}$$

The reversed formula reads

$$t = \frac{c}{b_0}\,\mathrm{Sin}\left(\frac{b_0}{c}t_0\right) = \frac{c}{2\,b_0}\left\{e^{\frac{b_0}{c}t_0} - e^{-\frac{b_0}{c}t_0}\right\} = t_0\left[1 + \frac{1}{3!}\left(\frac{b_0}{c}t_0\right)^2 + \frac{1}{5!}\left(\frac{b_0}{c}t_0\right)^4 + \dots\right]. \tag{54}$$

If one inserts this relation into Eqs. (47), (48), (49), and (51), then the acceleration, velocity, and displacement of the rocket are obtained as functions of the local time t_0 of the astronaut, namely

$$a = \frac{1}{\operatorname{Cos}\left(\frac{b_0}{c} t_0\right)} = 1 - \frac{1}{2!}\left(\frac{b_0}{c} t_0\right)^2 + \frac{5}{4!}\left(\frac{b_0}{c} t_0\right)^4 \mp \ldots \tag{55}$$

$$b = \ddot{x} = \frac{b_0}{\operatorname{Cos}^3\left(\frac{b_0}{c} t_0\right)} = b_0\left[1 - \frac{3}{2}\left(\frac{b_0}{c} t_0\right)^2 + \frac{11}{8}\left(\frac{b_0}{c} t_0\right)^4 \mp \ldots\right. \tag{56}$$

$$v = \dot{x} = c \operatorname{Tg}\left(\frac{b_0}{c} t_0\right) = b_0 t_0\left[1 - \frac{1}{3}\left(\frac{b_0}{c} t_0\right)^2 + \frac{2}{15}\left(\frac{b_0}{c} t_0\right)^4 \mp \ldots\right. \tag{57}$$

$$x = \frac{c^2}{b_0}\left[\operatorname{Cos}\left(\frac{b_0}{c} t_0\right) - 1\right] = \frac{b_0}{2} t_0^2\left[1 + \frac{1}{3 \cdot 4}\left(\frac{b_0}{c} t_0\right)^2 + \frac{1}{3 \cdot 4 \cdot 5 \cdot 6}\left(\frac{b_0}{c} t_0\right)^4 + \ldots\right]. \tag{58}$$

The relations collected in Table 2 thus hold for the motion of a rocket with constant self-acceleration b_0.

A rocket which travels with constant self-acceleration b_0 requires, in order to attain a given distance X, a time

$$T = \sqrt{\frac{2X}{b_0}} \tag{59}$$

in the system of classical physics (Euclidean time);

$$t = \frac{c}{b_0}\sqrt{\left(1 + \frac{b_0 X}{c^2}\right)^2 - 1} = \sqrt{\frac{2X}{b_0} + \left(\frac{X}{c}\right)^2} = T\sqrt{1 + \frac{1}{2}\left(\frac{b_0 X}{c^2}\right)} =$$
$$= T\left[1 + \frac{1}{2}\left(\frac{b_0 X}{2c^2}\right) - \frac{1}{2 \cdot 4}\left(\frac{b_0 X}{2c^2}\right)^2 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 6}\left(\frac{b_0 X}{2c^2}\right)^3 \mp \ldots\right] \tag{60}$$

in the system of the stationary earth observer;

$$t_0 = \frac{c}{b_0} \operatorname{Ar Cos}\left(1 + \frac{b_0 X}{c^2}\right) = T\,\frac{\operatorname{Ar Sin}\sqrt{\left(1 + \frac{b_0 X}{c^2}\right)^2 - 1}}{\sqrt{2\frac{b_0 X}{c^2}}} =$$
$$= T\left[1 - \frac{1}{2 \cdot 3}\left(\frac{b_0 X}{2c^2}\right) + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5}\left(\frac{b_0 X}{2c^2}\right)^2 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7}\left(\frac{b_0 X}{2c^2}\right)^3 \pm \ldots\right] \tag{61}$$

Table 2

Acceleration $b = \ddot{x} = \dot{v}$	$b_0 [1 - v^2/c^2]^{3/2}$	$\dfrac{b_0}{\left(1 + \dfrac{b_0 x}{c^2}\right)^3}$	$b_0 a^3$	$\dfrac{b_0}{\left[1 + \left(\dfrac{b_0}{c} t\right)^2\right]^{3/2}}$	$\dfrac{b_0}{\mathrm{Cos}^3 \left(\dfrac{b_0}{c} t_0\right)}$
Velocity $v = \dot{x}$	—	$c \dfrac{\sqrt{\left(1 + \dfrac{b_0 x}{c^2}\right)^2 - 1}}{1 + \dfrac{b_0 x}{c^2}}$	$c \sqrt{1 - a^2}$	$c \dfrac{\dfrac{b_0}{c} t}{\sqrt{1 + \left(\dfrac{b_0}{c} t\right)^2}}$	$c \, \mathrm{Tg} \left(\dfrac{b_0}{c} t_0\right)$
Displacement (or path) x	$\dfrac{c^2}{b_0}\left[\dfrac{1}{\sqrt{1 - v^2/c^2}} - 1\right]$	—	$\dfrac{c^2}{b_0}\left(\dfrac{1}{a} - 1\right)$	$\dfrac{c^2}{b_0}\left[\sqrt{1 + \left(\dfrac{b_0}{c} t\right)^2} - 1\right]$	$\dfrac{c^2}{b_0}\left[\mathrm{Cos}\left(\dfrac{b_0}{c} t_0\right) - 1\right]$
Contraction factor $a = \dfrac{dt_0}{dt}$	$\sqrt{1 - v^2/c^2}$	$\dfrac{1}{1 + \dfrac{b_0 x}{c^2}}$	—	$\dfrac{1}{\sqrt{1 + \left(\dfrac{b_0}{c} t\right)^2}}$	$\dfrac{1}{\mathrm{Cos}\left(\dfrac{b_0}{c} t_0\right)}$
Time t	$\dfrac{c}{b_0}\left[\dfrac{v/c}{\sqrt{1 - v^2/c^2}}\right]$	$\dfrac{c}{b_0}\sqrt{\left(1 + \dfrac{b_0 x}{c^2}\right)^2 - 1}$	$\dfrac{c}{b_0}\sqrt{\dfrac{1}{a^2} - 1}$	—	$\dfrac{c}{b_0} \mathrm{Sin}\left(\dfrac{b_0}{c} t_0\right)$
Proper time t_0	$\dfrac{c}{b_0} \mathrm{Ar\,Tg}\left(\dfrac{v}{c}\right)$	$\dfrac{c}{b_0} \mathrm{Ar\,Cos}\left(1 + \dfrac{b_0 x}{c^2}\right)$	$\dfrac{c}{b_0} \mathrm{Ar\,Cos}\left(\dfrac{1}{a}\right)$	$\dfrac{c}{b_0} \mathrm{Ar\,Sin}\left(\dfrac{b_0}{c} t\right)$	—
	$= \dfrac{c}{2 b_0} \ln \dfrac{1 + v/c}{1 - v/c}$	$= \dfrac{c}{b_0} \ln\left[1 + \dfrac{b_0 x}{c^2} + \sqrt{\left(1 + \dfrac{b_0 x}{c^2}\right)^2 - 1}\right]$	$= \dfrac{c}{b_0} \ln\left[\dfrac{1}{a} + \sqrt{\dfrac{1}{a^2} - 1}\right]$	$= \dfrac{c}{b_0} \ln\left[\dfrac{b_0}{c} t + \sqrt{1 + \left(\dfrac{b_0}{c} t\right)^2}\right]$	—

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in the system of the astronaut traveling with the rocket (proper or local time). In general $t_0 < T < t$. From this the immense gain in time from the relativistic principle can be recognized.

In order to simplify the numerical calculation it is of advantage to select c/b_0 as the unit of time, c as the unit of velocity, and c^2/b_0 as the unit of length. All relations can then be presented as dimensionless. With $c = 3 \cdot 10^{10}$ cm/sec and $b_0 = g = 981$ cm/sec^2

$$\frac{c}{g} = \frac{3 \cdot 10^{10}}{981} = 3{,}06 \cdot 10^7\,\mathrm{s} = 354{,}2\,\mathrm{d} = 0{,}97 \quad \text{(years)},$$

$$\frac{c^2}{g} = \frac{9 \cdot 10^{20}}{981} = 9{,}18 \cdot 10^{17}\,\mathrm{cm} = 9{,}18 \cdot 10^{12}\,\mathrm{km} = 0{,}97 \quad \text{(light years)}.$$

The thrust F_0 in the system of the astronaut is, according to Eq. (36),

$$m_0 b_0 = F_0 \equiv - a_0 \frac{dm_0}{dt_0}. \tag{62}$$

From this the law oı mass decrease is seen immediately to be

$$\frac{dm_0}{m_0} = - \frac{b_0}{a_0} dt_0$$

or, integrated, with constant outflow velocity a_0

$$\int_{M_0}^{m_0} \frac{dm_0}{m_0} = - \frac{b_0}{a_0} \int_0^{t_0} dt_0, \qquad \ln \frac{m_0}{M_0} = - \frac{b_0}{a_0} t_0,$$

that is upon use of Table 2

$$\frac{m_0}{M_0} = e^{-\frac{b_0}{a_0} t_0} = \left[\frac{b_0}{c} t + \sqrt{1 + \left(\frac{b_0}{c} t\right)^2}\right]^{-\frac{c}{a_0}}$$

$$= \left[1 + \frac{b_0 x}{c^2} + \sqrt{\left(1 + \frac{b_0 x}{c^2}\right)^2 - 1}\right]^{-\frac{c}{a_0}} = \left[\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}\right]^{-\frac{c}{2a_0}} \tag{63}$$

$$F_0 = m_0 b_0 = M_0 b_0 e^{-\frac{b_0}{a_0} t_0} = M_0 b_0 \left[\frac{b_0}{c} t + \sqrt{1 + \left(\frac{b_0}{c} t\right)^2}\right]^{-\frac{c}{a_0}} =$$

$$= M_0 b_0 \left[1 + \frac{b_0 x}{c^2} + \sqrt{\left(1 + \frac{b_0 x}{c^2}\right)^2 - 1}\right]^{-\frac{c}{a_0}} = M_0 b_0 \left[\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}\right]^{-\frac{c}{2a_0}} \tag{64}$$

By reversing Eq. (63) one finds

$$v = c \frac{1 - \left(\frac{m_0}{M_0}\right)^{\frac{2a_0}{c}}}{1 + \left(\frac{m_0}{M_0}\right)^{\frac{2a_0}{c}}}, \quad \text{thus} \quad \alpha = \sqrt{1 - v^2/c^2} = \frac{2\left(\frac{m_0}{M_0}\right)^{\frac{a_0}{c}}}{1 + \left(\frac{m_0}{M_0}\right)^{\frac{2a_0}{c}}}. \tag{65}$$

From this there follows for the displacement

$$x = \frac{c^2}{b_0}\left(\frac{1}{\alpha} - 1\right) = \frac{c^2}{b_0}\left[\frac{1 + \left(\frac{m_0}{M_0}\right)^{\frac{2a_0}{c}}}{2\left(\frac{m_0}{M_0}\right)^{\frac{a_0}{c}}} - 1\right] = \frac{c^2}{2b_0} \frac{\left[1 - \left(\frac{m_0}{M_0}\right)^{\frac{a_0}{c}}\right]^2}{\left(\frac{m_0}{M_0}\right)^{\frac{a_0}{c}}} \tag{66}$$

and for the time

$$t = \frac{c}{b_0}\left[\frac{v/c}{\alpha}\right] = \frac{c}{2b_0} \frac{1 - \left(\frac{m_0}{M_0}\right)^{\frac{2a_0}{c}}}{\left(\frac{m_0}{M_0}\right)^{\frac{a_0}{c}}}. \tag{67}$$

In contrast to this is, according to Eq. (63), the proper or local time

$$t_0 = -\frac{a_0}{b_0} \ln \frac{m_0}{M_0}. \tag{68}$$

IV. Motion of a Rocket with Constant Mass Drive (Thrust) in Free Space Without External Forces

In this case we again proceed from the equation of motion (36) of the rocket in the system of the astronaut traveling with the rocket, namely

$$\frac{m_0 \dot{v}}{(1 - v^2/c^2)^{3/2}} = m_0 b_0 = F_0 = - a_0 \frac{dm_0}{dt_0}; \tag{69}$$

In connection therewith the mass drive per second is now

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$$\mu_0 = - \frac{dm_0}{dt_0} = \text{const.} \tag{70}$$

The mass thus decreases linearly with the time t_0 according to the rule

$$m_0 = M_0 - \mu_0 t_0 = M_0 \left(1 - \frac{\mu_0}{M_0} t_0\right), \tag{71}$$

in which M_0 is the initial rest mass of the rocket at the time $t_0=0$. If we once more take the velocity of emanation of the gases a_0 =const., then the thrust also is

$$F_0 = a_0 \mu_0 = \text{const.} \tag{72}$$

On the other hand $F = F_0 (1 - v/a_0) = \mu_0(a_0 - v)$ is variable. The self-acceleration now advances according to the following rule:

$$\frac{\dot{v}}{(1 - v^2/c^2)^{3/2}} = b_0 = \frac{F_0}{m_0} = \frac{a_0 \mu_0}{M_0 - \mu_0 t_0} = a_0 \frac{\frac{\mu_0}{M_0}}{1 - \frac{\mu_0}{M_0} t_0} \tag{73}$$

Because $dt_0 = \sqrt{1 - v^2/c^2} \cdot dt$ there then holds

$$\frac{dv}{1 - v^2/c^2} = \frac{a_0 \mu_0}{M_0 - \mu_0 t_0} dt_0 = a_0 \frac{\frac{\mu_0}{M_0}}{1 - \frac{\mu_0}{M_0} t_0} dt_0$$

$$
= \frac{c}{2}\frac{M_0}{\mu_0}\left[\int\limits_{1-\frac{\mu_0}{M_0}t_0}^{1}\frac{dz}{z^{\frac{a_0}{c}}} - \int\limits_{1-\frac{\mu_0}{M_0}t_0}^{1} z^{\frac{a_0}{c}}\,dz\right] = \frac{c}{2}\frac{M_0}{\mu_0}\left[\frac{z^{1-\frac{a_0}{c}}}{1-\frac{a_0}{c}} - \frac{z^{1+\frac{a_0}{c}}}{1+\frac{a_0}{c}}\right]_{1-\frac{\mu_0}{M_0}t_0}^{1};
$$

consequently because of Eq. (74)

$$
\begin{aligned}
x &= \frac{c}{2}\frac{M_0}{\mu_0}\left[\frac{1-\left(\frac{m_0}{M_0}\right)^{1-\frac{a_0}{c}}}{1-\frac{a_0}{c}} - \frac{1-\left(\frac{m_0}{M_0}\right)^{1+\frac{a_0}{c}}}{1-\frac{a_0}{c}}\right] = \\
&= \frac{c}{2}\frac{M_0}{\mu_0}\left[\frac{1-\left(1-\frac{\mu_0}{M_0}t_0\right)^{1-\frac{a_0}{c}}}{1-\frac{a_0}{c}} - \frac{1-\left(1-\frac{\mu_0}{M_0}t_0\right)^{1+\frac{a_0}{c}}}{1+\frac{a_0}{c}}\right] = \qquad (79)\\
&= \frac{c}{2}\frac{M_0}{\mu_0}\left[\frac{1-\left(\frac{1-v/c}{1+v/c}\right)^{\frac{1}{2}\left(\frac{c}{a_0}-1\right)}}{1-\frac{a_0}{c}} - \frac{1-\left(\frac{1-v/c}{1+v/c}\right)^{\frac{1}{2}\left(\frac{c}{a_0}+1\right)}}{1+\frac{a_0}{c}}\right].
\end{aligned}
$$

In classical physics $(c \to \infty,\ \varepsilon \equiv a_0/c \to 0)$ formulas (76), (77), and (78) go over into

$$
a = \sqrt{1-v^2/c^2} = 1, \qquad b = b_0 = a_0\frac{\frac{\mu_0}{M_0}}{1-\frac{\mu_0}{M_0}t_0}, \qquad t = t_0.
$$

For Eqs. (75) and (79) a transition in the limit must be performed by means of the Bernoulli-de l'Hospital formula.

$$
\begin{aligned}
\frac{v}{a_0} &= \lim_{\varepsilon\to 0}\frac{1-\left(1-\frac{\mu_0}{M_0}t_0\right)^{2\varepsilon}}{\varepsilon\left[1+\left(1-\frac{\mu_0}{M_0}t_0\right)^{2\varepsilon}\right]} = \frac{1}{2}\lim_{\varepsilon\to 0}\frac{1-\left(1-\frac{\mu_0}{M_0}t_0\right)^{2\varepsilon}}{\varepsilon} = \\
&= \frac{1}{2}\lim_{\varepsilon\to 0}\left\{-2\left(1-\frac{\mu_0}{M_0}t_0\right)^{2\varepsilon}\ln\left(1-\frac{\mu_0}{M_0}t_0\right)\right\} = \qquad (80)\\
&= \ln\frac{1}{1-\frac{\mu_0}{M}t_0} = \ln\frac{M_0}{m_0}
\end{aligned}
$$

and

$$\frac{x}{a_0} = \frac{M_0}{\mu_0} \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \left[\frac{1 - \left(\frac{m_0}{M_0}\right)^{1-\varepsilon}}{1-\varepsilon} - \frac{1 - \left(\frac{m_0}{M_0}\right)^{1+\varepsilon}}{1+\varepsilon} \right] =$$

$$= \frac{M_0}{\mu_0} \lim_{\varepsilon \to 0} \frac{1}{1-\varepsilon^2} \left[1 - \frac{(1+\varepsilon)\left(\frac{m_0}{M_0}\right)^{1-\varepsilon} - (1-\varepsilon)\left(\frac{m_0}{M_0}\right)^{1+\varepsilon}}{2\varepsilon} \right] =$$

$$= \frac{M_0}{\mu_0} \lim_{\varepsilon \to 0} \left[1 - \frac{(1+\varepsilon)\left(\frac{m_0}{M_0}\right)^{1-\varepsilon} - (1-\varepsilon)\left(\frac{m_0}{M_0}\right)^{1+\varepsilon}}{2\varepsilon} \right] =$$

$$= \frac{M_0}{\mu_0} \lim_{\varepsilon \to 0} \left[1 - \frac{1}{2} \left\{ \left(\frac{m_0}{M_0}\right)^{1-\varepsilon} - (1+\varepsilon)\left(\frac{m_0}{M_0}\right)^{1-\varepsilon} \ln \frac{m_0}{M_0} + \right.\right.$$

$$\left.\left. + \left(\frac{m_0}{M_0}\right)^{1+\varepsilon} - (1-\varepsilon)\left(\frac{m_0}{M_0}\right)^{1+\varepsilon} \ln \frac{m_0}{M_0} \right\} \right] =$$

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$$= \frac{M_0}{\mu_0} \left[1 - \frac{m_0}{M_0} \left(1 - \ln \frac{m_0}{M_0} \right) \right] = \frac{M_0}{\mu_0} \left[1 - \left(1 - \frac{\mu_0}{M_0} t_0 \right) \cdot \left\{ 1 - \ln\left(1 - \frac{\mu_0}{M_0} t_0 \right) \right\} \right] = \tag{81}$$

$$= \frac{M_0}{\mu_0} \left[\frac{\mu_0}{M_0} t_0 + \left(1 - \frac{\mu_0}{M_0} t_0 \right) \ln \left(1 - \frac{\mu_0}{M_0} t_0 \right) \right].$$

Unfortunately it is not possible to express acceleration, velocity, and displacement also for the time t, as in the special case of constant self-acceleration, since Eq. (78) cannot be decomposed according to t_0. If one develops Eq. (78) in a power series

$$t = t_0 \left[1 + \left(\frac{a_0}{c}\right)^2 \frac{\left(\frac{\mu_0}{M_0} t_0\right)^2}{3!} + 3 \left(\frac{a_0}{c}\right)^2 \frac{\left(\frac{\mu_0}{M_0} t_0\right)^3}{4!} + \right.$$
$$\left. + \left(\frac{a_0}{c}\right)^2 \left\{ \left(\frac{a_0}{c}\right)^2 + 11 \right\} \frac{\left(\frac{\mu_0}{M_0} t_0\right)^4}{5!} + \ldots \right], \tag{82}$$

then the reversal reads

$$t_0 = t \left[1 - \left(\frac{a_0}{c}\right)^2 \frac{\left(\frac{\mu_0}{M_0} t\right)^2}{3!} - 3 \left(\frac{a_0}{c}\right)^2 \frac{\left(\frac{\mu_0}{M_0} t\right)^3}{4!} + \right.$$
$$\left. + \left(\frac{a_0}{c}\right)^2 \left\{ 9 \left(\frac{a_0}{c}\right)^2 - 11 \right\} \frac{\left(\frac{\mu_0}{M_0} t\right)^4}{5!} + \ldots \right]. \tag{83}$$

The values for t_0 ascertained by means of this series are now inserted into Eqs. (77), (75), and (79) in order to obtain the acceleration, velocity, and displacement of the rocket in their dependence on t.

All relations can be exhibited as dimensionless, if we choose M_0/μ_0

Index of Literature (Bibliography)

[1] E. Madelung, "Die mathematischen Hilfsmittel des Physikers" [The Mathematical Aids of the Physicist], Berlin: Springer, 415 (1950).

[2] J. Ackeret, "Zur Theorie der Raketen" [On the Theory of Rockets], Helv. Phys. Acta 19, 103 (1946); On the Theory of Rockets. J. Brit. Interplan. Soc., 6, 116 (1947).

[3] R. Esnault-Pelterie, "L'Astronautique" [Astronautics], Paris: A. Lahure, 229 (1930); "L'Astronautique" [Astronautics, Supplement], Paris: A. Lahure, 89 (1935).

[4] W. L. Bade, Relativistic Rocket Theory. Amer. Jour. Phys., 21, 310 (1953).

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